Global Connectivity by Timelike Geodesics

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Dedicated to Professor PASCUAL JORDAN *on the occasion of his 65th birthday*

The problem of existence of timelike or null geodetic arcs is examinated. It is shown that some transparent causality conditions are sufficient for the existence of an extremal geodetic arc between every pair of events joined by a timelike curve. In the case of a complete space-time these assumptions are also necessary. Some applications to the existence of global space-sections and the structure of space-times with a CAUCHY surface are given.

If an observer wants to explore the structure of space-time he must investigate the world-lines of test particles which are timelike or null geodesics in the concept of General Relativity. The question arises which events in space-time can be reached by these world-lines. It is a problem of particular interest whether the past or the future of an event is covered by the timelike (null) geodesics meeting this event.

From the mathematical point of view this problem can be formulated as follows:

Under what conditions can one conclude that two points in a normal-hyperbolic manifold joined by a timelike (null) curve can be joined by a timelike (null) geodesic?

The importance of such a statement is based on the fact that the family of all timelike curves is practically incalculable while the timelike and null geodesics form a rather surveyable set.

Definitions

Space-time is represented as a connected manifold with a metric tensor g_{ab} of signature $(-+++)$, and of differentiability class C^3.

A *curve* is a continuous mapping of a compact interval into space-time: $[0,1] \to V^4$ modulo a monotonic automorphism of $[0,1]$ i. e. modulo a parameter transformation.

A *c-curve* ("causal curve") is a timelike or null curve.

The *length* of a smooth curve g with respect to the metric tensor g_{ab} is given by the functional

$$F(g) := \int_{g(\tau)} \sqrt{|g_{ab}\, \dot{x}^a\, \dot{x}^b|}\, \mathrm{d}\tau\,.$$

A set $M \subset V^4$ is called *strongly causal* if every point in M has a neighbourhood U with the property that no c-curve which has left U can re-enter U.

A set $M \subset V^4$ is called *c-convex (strongly c-convex)* if every pair $A, B \in M$ joined by a c-curve can be joined by a c-geodesic (by a c-geodesic with a length not smaller than the length of any c-curve from A to B).

A space-time V^4 is called *c-complete* if every timelike or null geodesic can be extended to arbitrary values of an affine parameter.

§ 1. Summary of Statements on c-Convexity [1]

To find propositions on c-convexity and strong c-convexity one is compelled to investigate three sets:

- $\mathcal{H}(A,B)$ the set of all c-curves from A to B,
- $S(A,B)$ the set of all points met by some curve of $\mathcal{H}(A,B)$ i. e. the intersection of the future of A and the past of B,
- $\mathcal{G}_A$ the family of all c-geodesics through A.

The structure of $S(A,B)$ has been found to be essential for the subsequent examinations. $\mathcal{G}_A$ and $\mathcal{H}(A,B)$ have regular properties only if the following conditions are valid:

[1] For a concentrated discussion of the main results listed in this section, especially theorem 1, see [2]. Exhaustive proofs and detailed considerations of their physical meaning and applications are given in [3].

[2] H.-J. SEIFERT, Commun. Math. Phys. **4**, 324 [1967].

[3] H.-J. SEIFERT, Diplomarbeit, Hamburg 1967.

(A 1) *$S(A, B)$ is compact,* and
(A 2) *$S(A, B)$ is strongly causal.*

These conditions can be replaced by:

(A 1′) *Every c-geodesic through A leaves the past of B,* and (A 2).

If (A 1′) is not valid, the entire world-line of a test particle lies in the past of B. Thus the mathematically simple assumption (A 1) gets a physical significance (further details are given in § 2). Another pair of conditions equivalent to (A 1), (A 2) is:

(A 1) and
(A 2′) *In $S(A, B)$ there is no c-line that meets every neighbourhood of each of its points infinitely often.*

With these definitions we can state the propositions presented below. For a better understanding of the idea of proof accompanying each statement we begin with a discussion of the sets $\mathcal{H}(A, B)$ and $\mathcal{G}_A$.

For an accurate investigation of $\mathcal{H}$ and F it is not adequate to consider smooth curves only. Except for trivial cases the set of all smooth c-curves joining A and B is not compact, but the compactness is needed for a proof of existence of a maximum. Assume $\mathcal{H}$ as the closure of the smooth c-curves with respect to pointwise convergence. There are several possible topologies on $\mathcal{H}$. Here the topology of pointwise convergence in a suitable parametrisation is chosen. Any pointwise convergent sequence of c-curves is uniformly convergent with respect to some positive definite metric. If $S(A, B)$ obeys (A 1) and (A 2) then $\mathcal{H}(A, B)$ is compact, and has a finite number of connected components. These components are c-homotopy classes (explained in [4] and [3]).

The length F is an upper semi-continuous functional defined on the set of smooth c-curves. In order to define F on $\mathcal{H}$ assume F as the upper semi-continuous regularisation (smallest upper semi-continuous extension). F takes finite values only (§ 3 lemma 8). A (relative) maximum in $\mathcal{H}$ with respect to F is a geodetic arc.

It is a well known result that a space-time admits a covering by "simply convex" compact neighbourhoods U_i (see [5]); (each two points A, B in each U_i can be joined by a unique geodetic arc which lies in U_i). A c-geodetic arc in some U_i is F-maximal in its component of $\mathcal{H}$.

With these properties of $\mathcal{H}(A, B)$, $S(A, B)$, and $\mathcal{G}_A$ we claim the following statements on c-geodetic connectivity:

Theorem 1 (Closed world-lines are forbidden) [6]: *If $S(A, B)$ is not empty, compact, and strongly causal then there is a c-geodesic from A to B which assumes the maximal value of F on $\mathcal{H}(A, B)$; i. e.* (A 1) *and* (A 2) *imply strong c-convexity.*

Proof: As $\mathcal{H}(A, B)$ is compact and F is upper semi-continuous, an extremal curve must exist and must be a geodesic.

Proposition 2: *If $S(A, B)$ obeys* (A 1) *and* (A 2), *every connected component of $\mathcal{H}(A, B)$ contains a c-geodesic.*

Proof: Every component of the compact $\mathcal{H}$ is also compact.

Criterion 3: *If a* Cauchy *surface (see § 6) exists, the entire space-time is strongly c-convex* [7].

Proof: The existence of a Cauchy surface implies the validity of (A 1) and (A 2) for every $S(A, B)$ (see [2]).

Proposition 4 (The case of closed c-curves): *Assume that $S(A, B)$ is compact, and that there exists a connected component of $\mathcal{H}(A, B)$ such that every point has a neighbourhood which is not met more than n-times by any of its curves. Then this component contains a geodesic.*

Proof: This component is compact.

Criterion 5: *If $S(A, B)$ is compact and can be covered by spacelike hypersurfaces which have no boundary points in $S(A, B)$, then each component of $\mathcal{H}(A, B)$ contains a geodesic.*

Proof: Under this condition each component of $\mathcal{H}(A, B)$ fulfills the assumptions of proposition 4 [9].

Proposition 6: *If all c-geodesics through A leave the past of B, one of them must meet B.*

Proof: will be a consequence of the proof of Lemma 7 (§ 2).

[4] A. Avez, Thèses, Paris 1963.
[5] J. H. C. Whitehead, Quart. J. Math. **3**, 33 [1932].
[6] This theorem modifies a conjecture of Avez [4].
[7] See also Hawking [8] who gives a direct proof of the existence of an extremal geodetic connection in manifolds with a Cauchy surface.
[8] S. W. Hawking, Proc. Roy. Soc. London A **294**, 511 [1966].

[9] This criterion was stimulated by a suggestion of Beiglböck. Compare with the theorem of Ascoli which gives a condition for the compactness of a family of functions. The set of all functions g: $[0, 1] \to V^4$ is not compact with respect to pointwise convergence. In order to apply this theorem to a set of curves one must give a unique parametrisation. Such a possibility is equivalent to the validity of the assumptions made in criterion 5 if the covering is simple.

§ 2. Meaning of the Assumptions on $S(A, B)$

The consequences of the occurrence of closed world-lines are obvious: With regard to the finite accuracy in every measurement, there is not much difference between closed c-curves and the nearly closed c-lines mentioned in (A 2′).

It is more difficult to realize the significance of the compactness of $S(A,B)$. This question is answered by a lemma:

Lemma 7: *If $S(A,B)$ is incompact, then there exists a c-geodesic through A which remains in the past of B. Conversely: If there is an entire c-geodesic through A in the past of B, then $S(A,B)$ is incompact, or not strongly causal.*

Proof: The converse is evident. The first part of the lemma can be proved by contradiction. Assume every c-geodesic through A has a point on the boundary of the past of B. These points form a lightlike set L. A null geodetic generator of L must have endpoints; one is B, and the other is the last point in the past of B on a null geodesic through A. L is a compact connected part of the past light-cone of B. The union of all c-geodetic segments between A and L forms a *compact set* whose boundary consists of L, and a part of the future light-cone of A, and must therefore be *equal to* $S(A,B)$.

If $S(A,B)$ is incompact for some pair of events A, B, a c-geodesic g in the past of B exists, and can be interpreted as a world-line of a test particle started by an observer in A. If g is lightlike this particle can be a light-beam, if g is timelike it can be a rocket. Assume $S(A,B)$ strongly causal, then there are two possibilities:

(I) g cannot be extended to arbitrary values of an affine parameter. The test particle has a finite history; it must fall into a singularity.

(II) g exists for arbitrary values of an affine parameter. An observer moving along a c-curve from A to B sees the "ticking of a clock" along g accelerated boundlessly. Not later than at the arrival in B he has seen infinitely many periods (unlimited violet shift), and lost sight of the clock; i. e. g meets no longer his past light-cone. Conversely an observer on g sees the proceedings on the curve $\widehat{AB}$ retarded unlimitedly.

Clearly the statement of lemma 7 is time-symmetric: A c-geodesic g' through B lies in the future of A. It is possible that the observer along $\widehat{AB}$ sees new particles within his horizon. These must emerge from a singularity, or come from infinite distance. During their entire lifetime these particles can be influenced by the event A.

§ 3. A Necessary Condition

A necessary condition for the existence of a maximal c-geodesic joining A and B is:

$$\text{(B)} \quad \sup_{\widehat{AB} \in \mathscr{H}(A,B)} F(\widehat{AB}) < \infty \,.$$

This is established in the following lemma:

Lemma 8: *Every c-arc segment has finite length.*

Proof: A c-arc is given by a mapping g: $[0,1] \to V^4$. Take a covering of the image of g with simply convex neighbourhoods U_i (see § 1). As $[0,1]$ is compact, the image of g meets every U_i at most finitely often. Consequently one can find a c-curve g' consisting of a finite number of c-geodetic arcs with the property:

$$\infty > F(g') \geqq F(g) \,.$$

If the condition (B) is not valid for some $\mathcal{H}(A,B)$ then there must be a particle, e. g. a rocket started in A with a timelike world-line which stays in the past of B for every value of the proper time; compare with § 2 lemma 7:

If (A 1) is not valid there must be a timelike, or null geodetic line in the past of B; if (B) is not valid there exists a timelike, not necessarily geodetic line in the past of B.

The following lemma disproves the validity of (B) in c-complete space-times in some important cases:

Lemma 9: *Assume A lying in a c-complete space-time, and a null geodesic l meeting B in the future of A. If l does not meet A, there exist c-curves of arbitrary length between A and the points on l.*

Proof: Take a one-parameter set l_u ($u \in [0,1]$) of null geodesics embedding l which start from a c-curve between A and B with continuously differentiable directions. They form a 2-surface E. E cannot branch because a sequence of geodesics converging in a small interval of an affine parametrisation converges for every value of the affine parameter. E is everywhere timelike: Assume that E is lightlike in some point P on l_u, then the tangent vector k^a of l_u is orthogonal on a deviation vector η^a to a neighbouring geodesic $l_{u+\mathrm{d}u}: k^a \eta_a = 0$. This equation must hold along all of l_u, in contradiction to the existence of two c-directions in the starting-point. In the same way, the occurrence of a caustic can be disproved: $\eta^a = 0$ implies $k^a \eta_a = 0$. There are essentially three cases with regard to the behaviour of E at infinity. A coordinate transformation of a two-dimensional manifold E can change the infinite distance into a boundary with infinite g_{ab}. If the boundary near the endpoint of l is lightlike there is no big difference to MINKOWSKI space-time, and the result of this lemma is calculable. If the boundary is timelike there are c-curves along the boundary with unlimited length; if the boundary is spacelike one can find a timelike geodesic which falls into the endpoint of l, and must have infinite length because of the c-completeness.

Corollary: Consider a c-complete space-time. If the past of B contains an entire c-geodesic g through some A' then for every A in the past of A' not met by g we have: $\sup F(\widehat{AB}) = \infty$.

Proof: In the case of a null geodesic this is shown in lemma 9; for a timelike geodesic this is evident even if A is equal to A', or l meets A.

§ 4. Examination of the Necessity of the Conditions on S(A, B)

In this section only c-complete space-times are considered. It is shown that the assumptions (A 1) and (A 2) which suffice for strong causality are also essentially necessary.

Theorem 10: *Be A, B a pair of points joined by a c-curve. For the existence of a maximal c-geodetic arc $\overline{AB}$ it is necessary that* (A 1) *and* (A 2) *are valid for every pair A', B' in the interior of $S(A,B)$.*

Proof: Assume an incompact $S(A', B')$. Lemma 7 proves the existence of a c-geodesic in the past of B', and lemma 9 (corollary) shows that $\sup F(\widehat{AB'}) < \infty$ cannot be valid. If (A 2′) is not valid, one must distinguish two cases: (I) A closed c-curve g (extensible to a c-curve $\widehat{AB}$) with $F(g) > 0$ exists; considering sufficiently many circulations one gets: $\sup F(\widehat{AB}) = \infty$. (II) A closed or nearly closed null geodesic exists; lemma 9 shows: $\sup F(\widehat{AB}) = \infty$.

Corollary: Consider a maximal region M with validity of (A 1) and (A 2), i. e. no N with (A 1) and (A 2) includes M as a proper subset. If one extends M to a strongly causal region M', and to a region M'' which satisfies the necessary condition (B) for all A, $B \in M''$, this extension cannot reach beyond the closure of M: $\overline{M} \supset M''$, $\overline{M} \supset M'$.

By the way: M includes $S(A, B)$ if A, $B \in M$, and has a lightlike boundary.

Corollary: *A c-complete space-time is strongly c-convex, if and only if it satisfies either* (A 1) *and* (A 2) *or* (B) *for all pairs A, B.*

A similar discussion of the assumptions for proposition 4, and for incomplete space-times is given in [3].

§ 5. The Future of an Event

In this section only strongly causal, and time-oriented space-times are considered. Under these conditions the future $Z(A)$ of an event A is defined, and is a proper subset of the manifold. For an investigation of Z it is suitable to consider two subsets:

Z^f: The part of Z with a finite distance from A; i. e. $P \in Z^f$ if and only if $\sup F(\widehat{AP}) < \infty$, $P \in Z$.

Z^g: The part of Z reached by the c-geodesics through A.

Z^g is covered by the *transversal hypersurfaces*

$$C_\tau := \{P \mid \exists\, \overline{AP} \in \mathcal{G}_A, \quad F(\overline{AP}) = \tau\}.$$

Except C_0 (the light-cone), all C_τ's are spacelike where smooth. Z^f is covered by the pseudospheres

$$D_\tau := \{P \mid \sup F(\widehat{AP}) = \tau\}.$$

Theorem 10 shows that the interior of Z^f is contained in Z^g. D_τ has no boundary points in the interior of Z^f, and must be spacelike. D_τ is a part of C_τ; if D_τ is a proper part of C_τ, the boundary between D_τ and $C_\tau - D_\tau$ must consist of self-intersections of C_τ. If $Z^f(A) = Z(A)$, every D_τ is a slice, i. e. spacelike hypersurface without boundary points. Hence:

Proposition 11: *If for some point A there exists no point B obeying $\sup F(\widehat{AB}) = \infty$ then there exists a slice. Especially, if the space-time V^4 satisfies the causal conditions* (A 1) *and* (A 2) *for every pair A, B, V^4 is (not necessarily simply) covered by slices.*

In the recent developments of the global theory of space-time the existence of slices plays a rather important rôle (see [8]) but the causal meaning of such an assumption is not yet understood. This proposition proves the existence of a slice under strong causal conditions. It seems plausible that the occurrence of slices is provable under much weaker requirements.

§ 6. Cauchy Surfaces

Definition: A set $H \subset V^4$ is called CAUCHY *surface* if every not extensible c-line has exactly one point of intersection with H.

A time-oriented space-time V^4 with a CAUCHY surface H has a rather transparent topological and causal structure: The conditions (A 1) and (A 2) are valid for every pair A, B so that V^4 is strongly c-convex (criterion 3):

$$Z(A) = Z^{\mathrm{f}}(A) = Z^{\mathrm{g}}(A)$$

for every point in V^4. Moreover:

Proposition 12: *If a c-complete, time-oriented space-time V^4 admits a* CAUCHY *surface H, then there exists a homeomorphism f: $H \times (-\infty, +\infty) \rightarrow V^4$ such that $f(H \times \{\tau\}) =: D_\tau$ is a* CAUCHY *surface for every real number τ* [10].

Proof: The definitions in the last sections can be transposed to the future of H. The D_τ's ($\tau \geqq 0$) cover the future of H, while the D_τ's ($\tau \leqq 0$) cover the past.

Let us show that these D_τ's form CAUCHY surfaces: Consider a c-geodesic l which must meet H in some point P. Lemma 9 (corollary) shows that from a point P' in the past (future if $\tau < 0$) of P there are c-arcs joining P' and l with a length greater than $|\tau| + \sup F(\widehat{P'P})$, consequently l must intersect D_τ. The c-geodetic segments between an arbitrary point A and D_τ form a cone $S(A, D_\tau)$. As $S(A, D_\tau)$ is compact, every c-line g through A sufficiently extended must leave $S(A, D_\tau)$; as V^4 is time-oriented, g must leave $S(A, D_\tau)$ in a point of D_τ.

Any two CAUCHY surfaces are homeomorphic: Any time-oriented V^4 admits a continuously differentiable timelike vector-field whose integral curves define a homeomorphism.

The assumption of time-orientation is essential. As a counterexample consider the MÖBIUS band defined as flat space-time V^2: $\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}t^2$ with the identification of $(0, t)$ and $(1, -t)$. V^2 has a CAUCHY surface: $\{t=0\}$; every CAUCHY surface intersects $\{t=0\}$, and a point (x, t) cannot lie on a CAUCHY surface if $|t| > 1$.

Although the occurrence of a CAUCHY surface is a strong restriction on the topological and causal properties, the causal structure is not defined uniquely. For instance one can construct the following space-time of "trouser type" [3]: There are pairs of events A, B with a non-empty intersection of the future of A and the future of B but an empty intersection of the past of A and the past of B. The "legs" can be separated by a slice but not by a CAUCHY surface (see [11]). V^4 is time-oriented and c-complete, and admits a CAUCHY surface homeomorphic to R^3, but it has not the causal structure of MINKOWSKI space-time.

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[10] This is a supplement to a result of GEROCH and PENROSE who showed that a time-oriented, not necessarily c-complete V^4 with a CAUCHY surface H is the topological product $H \times (-\infty, +\infty)$ (private communication to W. KUNDT).

[11] W. KUNDT, Commun. Math. Phys. **4**, 143 [1967].